Geometric Aspects of Quasi-Periodic Property of Dirichlet Functions

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Introduction

By a general Dirichlet series we understand an expression of the form

$$\zeta_{A,\Lambda}(s) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n s}, \tag{1}$$

where $A = (a_n)$ is an arbitrary sequence of complex numbers and $\lambda_1 < \lambda_2 < \ldots$ is an increasing sequence of non negative numbers with $\lim_{n\to\infty} \lambda_n = \infty$.

We will consider only normalized series (1) in which $a_1 = 1$ and $\lambda_1 = 0$.

It is known [5] that if the series (1) converges for $s = s_0 = \sigma_0 + it_0$, then it converges for every s with $\operatorname{Re} s > \sigma_0$.

$$\zeta_{A,\Lambda}(s) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n s},$$

The number $\sigma_c = \inf\{\sigma \mid \zeta_{A,\Lambda}(\sigma) \text{ converges}\}$, when it exists, is called the **abscissa of convergence** of the series (1). When the series does not converge for any $s \in \mathbb{C}$ we denote $\sigma_c = +\infty$ and if it converges for every s we put $\sigma_c = -\infty$. The line $\operatorname{Re} s = \sigma_c$ is called the **line of convergence** of (1), although there are examples of Dirichlet series (see [5]) which do not converge for any s with $\operatorname{Re} s = \sigma_c$. Other series converge for all the points of that line, or only for some points.

When

$$\zeta_{A,\Lambda}(s) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n s},$$

does not converge for $s = 0$, then (see [5])

$$\sigma_c = \limsup_{n\to\infty} \frac{1}{\lambda_n} \log \left| \sum_{k=1}^{n} a_k \right| \geq 0 \tag{2}$$

If (1) converges for $s = 0$, then

$$\sigma_c = \limsup_{n\to\infty} \frac{1}{\lambda_{n+1}} \log \left| \zeta_{A,\Lambda}(0) - \sum_{k=1}^{n} a_k \right| \tag{3}$$

The abscissa σ_a of absolute convergence of the series (1) is defined in an analogous way and it is obvious that $-\infty \leq \sigma_c \leq \sigma_a \leq +\infty$. For the Riemann Zeta function $\sigma_c = \sigma_a = 1$, while for the alternate Zeta function $\sigma_c = 0$ and $\sigma_a = 1$. When $\sigma_c < +\infty$ and $\zeta_{A,\Lambda}(s)$ converges uniformly on compact sets of the half plane $\operatorname{Re} s > \sigma_c$ then $\zeta_{A,\Lambda}(s)$ is an analytic function in that half plane and sometimes it can be continued analytically to the whole complex plane except possibly for some poles. We keep the notation $\zeta_{A,\Lambda}(s)$ for this extended function and we call it Dirichlet function. Since $a_1 = 1$ and $\lambda_1 = 0$ in the series (1) we have that $\lim_{\sigma \to +\infty} \zeta_{A,\Lambda}(\sigma + it) = 1$ and it can be easily seen [2] that this limit is uniform with respect to t, in other words for every $\varepsilon > 0$ there is σ_ε such that for $\sigma > \sigma_\varepsilon$ we have

$$|\zeta_{A,\Lambda}(\sigma + it) - 1| < \varepsilon$$

for every real t. In particular, if σ_a is finite then there is $\alpha > 0$ such that the series (1) converges uniformly in the half plane

$\operatorname{Re} s > \sigma_a + \alpha$. Indeed, we can take $\sigma_\varepsilon > \sigma_a$ to make sure that $\sum_{n=1}^{\infty} |a_n| e^{-\lambda_n \sigma_\varepsilon}$ converges and let $\alpha = \sigma_\varepsilon - \sigma_a$, $s = \sigma + it$, where $\sigma \geq \sigma_\varepsilon$. Then, for every natural number N we have

$$\begin{aligned}\left|\sum_{n=N}^{\infty} a_n e^{-\lambda_n s}\right| &\leq \sum_{n=N}^{\infty} |a_n| e^{-\lambda_n \sigma} = e^{-\lambda_N (\sigma - \sigma_\varepsilon)} \sum_{n=N}^{\infty} |a_n| e^{-(\lambda_n - \lambda_N)(\sigma - \sigma_\varepsilon)} e^{-\lambda_n \sigma_\varepsilon} \\ &\leq e^{-\lambda_N \alpha} \sum_{n=1}^{\infty} |a_n| e^{-\lambda_n \sigma_\varepsilon},\end{aligned}$$

which tends to zero as $N \to \infty$, uniformly with respect to s. We can define the abscissa of uniform convergence of (1) as $\sigma_u = \inf\{\sigma | \zeta_{A,\Lambda}(\sigma + it) \text{ converges uniformly}\}$.

Studying Dirichlet L-functions $f(s)$ generated by ordinary Dirichlet series (the case where $\lambda_n = \log n$) Harald Bohr (see [3]) discovered that they display on vertical lines a quasi-periodic behavior, namely for every bounded domain Ω of uniform convergence of the series and for every $\varepsilon > 0$ there is a sequence (τ_n) ,

$$\begin{aligned} \dots \tau_{-2} < \tau_{-1} < 0 < \tau_1 < \tau_2 < \dots \\ \liminf_{n\to\pm\infty}(\tau_{n+1} - \tau_n) > 0, \\ \limsup_{n\to\pm\infty} \frac{\tau_n}{n} < \infty \end{aligned} \tag{4}$$

such that for every $s \in \Omega$ we have $|f(s + i\tau_n) - f(s)| < \varepsilon$.

This roughly means that the function comes (quasi)-periodically on a vertical line as close as we want to the value of it at any point of Ω belonging to that line. We study in this paper the quasi-periodic property of functions defined by general Dirichlet series and show that this is a geometric property of the image by $\zeta_{A,\Lambda}(s)$ of vertical lines related to the fundamental domains of these functions.

The Quasi-Periodicity on Vertical Lines of General Dirichlet Series

Let us give first to the concept of quasi-periodicity a slightly different definition. We will say that $f(s)$ is quasi-periodic on a line $\operatorname{Re} s = \sigma_0$ if for every $\varepsilon > 0$ and for every $s = \sigma_0 + it$ a sequence (4) exists such that $|f(s + i\tau_n) - f(s)| < \varepsilon$. We notice that this definition is no more attached to bounded domains, hence it appears less restrictive than that given by Bohr, yet the inequality refers only to the points of a given vertical line and not to the points of any vertical line intersecting the domain Ω, which is a restriction. This new definition serves better the purpose of studying the denseness properties of Dirichlet functions.

Theorem

If λ_n with $n = 2, 3, \ldots$ are linearly independent in the field of rational numbers then the series

$$\zeta_{A,\Lambda}(s) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n s},$$

is quasi-periodic on every vertical line of the half plane $\operatorname{Re} s > \sigma_u$.

Proof.
Let s be arbitrary with $\operatorname{Re} s = \sigma_0 > \sigma_u$ and divide $\zeta_{A,\Lambda}(s)$ and $\zeta_{A,\Lambda}(s + i\tau)$ into the sum A_n of the first n terms and the rest R_n. Since the series converges uniformly on $\operatorname{Re} s = \sigma_0$, when $\varepsilon > 0$ is given, there is a rank n such that $|R_n(s)| < \dfrac{\varepsilon}{3}$ and $|R_n(s + i\tau)| < \dfrac{\varepsilon}{3}$ for every real number τ. On the other hand

$$|A_n(s + i\tau) - A_n(s)| = \left|\sum_{k=1}^{n} a_k e^{-\lambda_k s}(e^{-i\lambda_k \tau} - 1)\right| \tag{5}$$

By Diophantine approximation, a sequence (4) exists such that for every τ_m of that sequence $e^{-i\lambda_k \tau_m}$ is as close to 1 on the unit circle as we whish. Since the set $\{a_k e^{-\lambda_k s}\}$ is bounded, we have $|A_n(s + i\tau) - A_n(s)| < \dfrac{\varepsilon}{3}$ for every $\tau = \tau_m$ and then $|\zeta_{A,\Lambda}(s + i\tau) - \zeta_{A,\Lambda}(s)| < \varepsilon$ for every $\tau = \tau_m$, which proves the theorem. □

For ordinary Dirichlet series we have $\lambda_n = \log n$, and for $n = 2, 3, \ldots$ they are linearly independent in the field of rational numbers, therefore these series are quasi-periodic on every vertical line from the half plane of convergence.

It is known (see [7]) that for every series

$$\zeta_{A,\Lambda}(s) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n s},$$

which can be continued analytically to a the whole complex plane except possibly for a simple pole at $s = 1$, the complex plane is divided into infinitely many horizontal strips S_k, $k \in \mathbb{Z}$ bounded by components of the pre-image of the real axis which are mapped bijectively by $\zeta_{A,\Lambda}(s)$ onto the interval $(1, \infty)$.

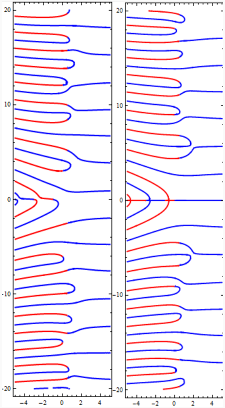
Strips S_k I

These are unbounded curves Γ'_k such that for $\sigma + it \in \Gamma'_k$ we have

$$\lim_{\sigma \to +\infty} \zeta_{A,\Lambda}(\sigma + it) = 1$$

and no Γ'_k can be contained in a right half plane. If S_k, $k \neq 0$ contains j_k zeros of $\zeta_{A,\Lambda}(s)$ counted with multiplicities then it will contain $j_k - 1$ zeros of $\zeta'_{A,\Lambda}(s)$.

Strips S_k



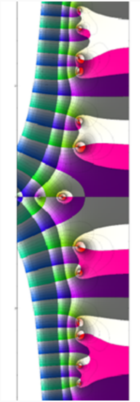


Figure 1:

For any Dirichlet function $\zeta_{A,\Lambda}(s)$ every vertical line which does not pass through the pole is divided by the boundaries of the fundamental domains $\Omega_{k,j}$ into finite intervals which are mapped bijectively by $\zeta_{A,\Lambda}(s)$ onto Jordan arcs $\gamma_{k,j}$. The image of the whole line by $\zeta_{A,\Lambda}(s)$ is therefore the union of infinitely many Jordan arcs $\gamma_{k,j}$.

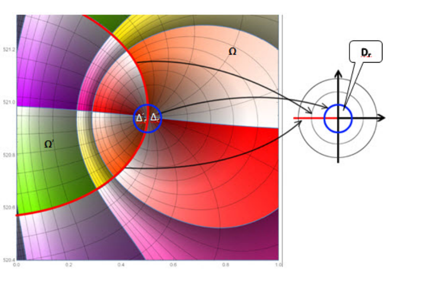


Figure 2:

If two domains $\Omega_{k,j}$ are adjacent, then the ends of the corresponding $\gamma_{k,j}$ which are images of the same point of the vertical line will obviously coincide. Moreover, different arcs $\gamma_{k,j}$ can have some other common points. The image of an interval determined by S_k is a bounded curve starting and ending on $(0, \infty)$ and having a finite number of intersections with the real axis and also a finite number of self intersection points. These last points represent the intersections of different arcs $\gamma_{k,j}$ as well as points corresponding to zeros of $\zeta'_{A,\Lambda}(s)$. At the points which are zeros of $\zeta'_{A,\Lambda}(s)$ the corresponding arcs are tangent to each other (see [6]). If the vertical line passes through the pole, then the images of the segments starting or ending at the pole are unbounded curves.

When the analytic continuation to the whole complex plane of the series (1) is possible the arcs $\gamma_{k,j}$ are defined for any vertical line, not only for the lines included in the half plane of convergence of this series. Then, expressing the quasi-periodic property in terms of these arcs, we can extend this concept to any vertical line. The extension can be performed by noticing that if $\zeta_{A,\Lambda}(s)$ is quasi-periodic on a vertical line $\operatorname{Re} s = \sigma_0$ from the half plane of convergence of (1) then for every $\varepsilon > 0$ and every s with $\operatorname{Re} s = \sigma_0 > \sigma_c$ there is a sequence (4) such that

$$|\zeta_{A,\Lambda}(\sigma_0 + it) - \zeta_{A,\Lambda}(s)| < \varepsilon \tag{6}$$

for $|t - \tau_m| > 0$ small enough where τ_m is any term of the sequence (4). This means that to every term τ_m of this sequence corresponds an arc $\gamma_{k,j}$ such that the point $\zeta_{A,\Lambda}(\sigma_0 + it)$ on that arc is located at a distance less than ε of $\zeta_{A,\Lambda}(s)$ for $|t - \tau_m|$ small enough. Then we can say that $\zeta_{A,\Lambda}(s)$ is quasi-periodic on the arbitrary line $\operatorname{Re} s = \sigma_0$ (not necessarily belonging to the half plane of uniform convergence of this series) if for every s with $\operatorname{Re} s = \sigma_0$ and every $\varepsilon > 0$ there are infinitely many fundamental domains $\Omega_{k,j}$ of $\zeta_{A,\Lambda}(s)$ such that the inequality (6) is satisfied for some t with $\zeta_{A,\Lambda}(\sigma_0 + it) \in \gamma_{k,j}$.

Analytic Continuation of General Dirichlet Series

It is known that some functions defined by Dirichlet series cannot be extended across the line $\operatorname{Re} s = \sigma_c$ since all the points of the abscissa of convergence are singular points. Such series can be easily found as seen in [5] and [2]. On the other hand all the Dirichlet L-functions are analytic continuations to the whole complex plane, except for some poles of particular Dirichlet series. These continuations have been performed by using the Riemann technique of contour integration. We will show next that a similar technique is applicable also to general Dirichlet series.

We recall that the Gamma function can be expressed as:

$$\Gamma(s) = \int_0^{\infty} x^{s-1} e^{-x} dx \tag{7}$$

and this is a meromorphic function in the complex plane.

Theorem

Suppose that the series

$$\zeta_{A,\Lambda}(s) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n s},$$

has a finite abscissa of convergence σ_c and it is convergent on $\operatorname{Re} s = \sigma_c$ except for some isolated points. Then if the integral of $(-z)^{s-1}\zeta_{A,e^\Lambda}(z)$ on a half circle C_r centered at the origin and of radius r situated in the right half plane tends to zero as $r \to 0$ the series (1) can be extended to a meromorphic function in the whole complex plane.

On replacing x by $e^{\lambda_n}x$ in (7) we obtain

$$e^{-\lambda_n s} = \frac{1}{\Gamma(s)} \int_0^\infty x^{s-1} e^{-e^{\lambda_n} x} dx,$$

which multiplied by a_n and added gives

$$\zeta_{A,\Lambda}(s) = \frac{1}{\Gamma(s)} \int_0^\infty x^{s-1} \zeta_{A,e^\Lambda}(x) dx \tag{8}$$

Here we have denoted by e^Λ the sequence $e^{\lambda_1}, e^{\lambda_2}, \ldots$ and we have interchanged the integration and the summation, which is allowed, since the integrals of the terms are absolutely convergent at both ends.

We notice that Hardy and Riesz [5] found (Theorem 11) a similar formula to (8) for $\operatorname{Re} s > 0$, yet they did not use it to extend the function $\zeta_{A,\Lambda}(s)$. For the integral in (8) to exist, the function $\zeta_{A,e^\Lambda}(x)$ must be defined at least for $x > 0$, which is the case since if

$$\limsup_{n\to\infty} \frac{1}{\lambda_n} \log\left|\sum_{k=1}^{n} a_k\right|$$

is finite, then

$$\limsup_{n\to\infty} \frac{1}{e^{\lambda_n}} \log\left|\sum_{k=1}^{n} a_k\right| = 0.$$

Also, if

$$\limsup_{n\to\infty} \frac{1}{\lambda_{n+1}} \log \left| \zeta_{A,\Lambda}(0) - \sum_{k=1}^{n} a_k \right|$$

is finite, then

$$\limsup_{n\to\infty} \frac{1}{e^{\lambda_{n+1}}} \log \left| \zeta_{A,e^\Lambda}(0) - \sum_{k=1}^{n} a_k \right| = 0.$$

Yet, for $\sigma_c \geq 0$ the series $\zeta_{A,\Lambda}(0) = \zeta_{A,e^\Lambda}(0) = \Sigma_{n=1}^{\infty} a_n$ is divergent and the integrand in (8) can be divergent too at 0.

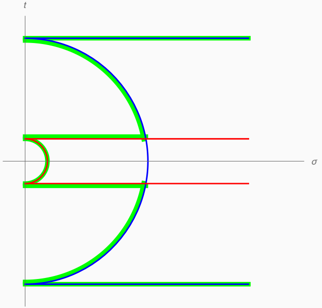
This happens, for example, for the Riemann Zeta function, for which $\sigma_c = 1$ (see [1]) and (8) becomes

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{e^x - 1} dx \tag{8a}$$

with $\lim\limits_{x \searrow 0} \dfrac{1}{e^x - 1} = +\infty$.

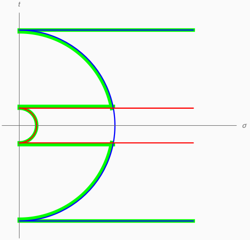
However, the integral (8a) converges absolutely and has been evaluated by contour integration. Using the contour C (see [1], p. 216) in the case of $\zeta(s)$ has been possible since the integrand $\dfrac{(-z)^{s-1}}{e^z - 1}$ is defined on it. Yet, for a general Dirichlet series $\zeta_{A,\Lambda}(s)$ the integrand $(-z)^{s-1}\zeta_{A,e^\Lambda}(z)$ might have no meaning at the left of the imaginary axis.

There is however a way to circumvent this situation, namely to form a similar contour γ_r but with a half circle C_r centered at the origin and of radius r situated in the right half plane, as seen in Fig. **??** below. Here γ_r is the contour formed by a small half circle $C_r : z = re^{i\theta}$, $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ and by the half lines $\operatorname{Im} z = r$, $\operatorname{Re} z \geq 0$ and $\operatorname{Im} z = -r$, $\operatorname{Re} z \geq 0$.



If γ_R is the blue contour, γ_r the red one and γ the green contour, then

$$\int_{\gamma_r} (-z)^{s-1} \zeta_{A,\Lambda}(z) dz = \int_{\gamma_R} (-z)^{s-1} \zeta_{A,\Lambda}(z) dz$$



since both of these integrals are obviously equal to $\int_\gamma (-z)^{s-1} \zeta_{A,\Lambda}(z) dz$ therefore the integral on γ_r does not really depend on r.

In order for the extension of $\zeta_{A,\Lambda}(z)$ to a meromorphic function in the whole complex plane to be possible it is sufficient that the integral on C_r of $(-z)^{s-1}\zeta_{A,e^{\Lambda}}(z)$ tends to zero as $r \to 0$. In such a case the analytic continuation of $\zeta_{A,\Lambda}(s)$ is given by the formula

$$\zeta_{A,\Lambda}(s) = -\frac{\Gamma(1-s)}{2\pi i}\int_{\gamma_r}(-z)^{s-1}\zeta_{A,e^{\Lambda}}(z)dz \tag{9}$$

which is equivalent with (8). Indeed, $(-z)^{s-1}$ is unambiguously defined as $e^{(s-1)\log(-z)}$ and if the integral in (9) on the half circle C_r tends to zero as $r \to 0$, then using the argument from [1], page 214, we see that (9) and (8) are equivalent. The right hand side in (8) is defined for every complex value s and represents a meromorphic function in the whole complex plane.

Quasi-Periodicity and Denseness Property

The connection between the quasi-periodic property and the denseness property of the image of vertical lines by Dirichlet functions appears clearly when we interpret the first one in terms of the arcs $\gamma_{k,j}$. Indeed, we can prove the following:

Theorem

The necessary and sufficient condition for $\zeta_{A,\Lambda}(s)$ to be quasi-periodic on the line $\operatorname{Re} s = \sigma_0$ is that for for every $\varepsilon > 0$ and every point s on that line infinitely many fundamental domains $\Omega_{k,j}$ exist such that the corresponding arcs $\gamma_{k,j}$ intersect the disc $|z - \zeta_{A,\Lambda}(s)| < \varepsilon$.

Proof I

The condition is necessary, since if for every $\varepsilon > 0$ and every s on the line $\operatorname{Re} s = \sigma_0$ a sequence (4) exists with the property that $|\zeta_{A,\Lambda}(s + i\tau_n) - \zeta_{A,\Lambda}(s)| < \varepsilon$, then the number of domains $\Omega_{k,j}$ containing points $s + i\tau_n$ must be infinite. Indeed, the inequality $\liminf\limits_{n\to\pm\infty}(\tau_{n+1} - \tau_n) > 0$ implies that every $\Omega_{k,j}$ can contain only a finite number of points $s + i\tau_n$. Then there must be infinitely many arcs $\gamma_{k,j}$ intersecting the disc $|z - \zeta_{A,\Lambda}(s)| < \varepsilon$. Vice versa, if infinitely many such arcs exist, then choosing a point $s + i\tau_n$ belonging to that disc on each one of them, the sequence (τ_n) satisfies obviously the conditions (4).

Hence, if $\zeta_{A,\Lambda}(s)$ is quasi-periodic on the line $\mathrm{Re}\, s = \sigma_0$ then any small neighborhood V of every point $\zeta_{A,\Lambda}(s)$ with $\mathrm{Re}\, s = \sigma_0$ intersects infinitely many arcs $\gamma_{k,j}$, therefore the image of $\mathrm{Re}\, s = \sigma_0$ is dense in (at least a part of) V. Indeed, there is an arc γ_{k_0,j_0} passing through $\zeta_{A,\Lambda}(s)$ and as close as we want of this arc pass infinitely many arcs $\gamma_{k,j}$, which are parts of the image of the same line and every one of these arcs has the same property, hence the denseness of the image of $\mathrm{Re}\, s = \sigma_0$ in the part of V around this arc is obvious. Then we can say that the quasi-periodic property of $\zeta_{A,\Lambda}(s)$ on the line $\mathrm{Re}\, s = \sigma_0$ implies the denseness in itself of the image of that line by $\zeta_{A,\Lambda}(s)$. Indeed, every point of γ_{k_0,j_0} is either an intersection point with another arc $\gamma_{k,j}$ or a tangent point with such an arc or a limit point of arcs $\gamma_{k,j}$ in the sense that there is a sequence of such arcs whose distance to that point tends to zero.

No two arcs $\gamma_{k,j}$ and $\gamma_{k',j'}$ can overlap partially. Indeed, in the contrary case, we would have that the conformal mapping of $\Omega_{k,j}$ onto $\Omega_{k',j'}$ given by $f|_{\Omega_{k',j'}}^{-1} \circ f(s)$, where $f(s) = \zeta_{A,\Lambda}(s)$, maps an interval of $\operatorname{Re} s = \sigma_0$ onto another interval of the same line, which would be possible only if f(s) were a linear transformation and this is not the case. Hence the image of the line $\operatorname{Re} s = \sigma_0$ by $\zeta_{A,\Lambda}(s)$ cannot have arcs towards which no other arcs accumulate. Its closure is necessarily a two dimensional set whose boundary is formed by arcs $\gamma_{k,j}$ or limit points of such arcs.

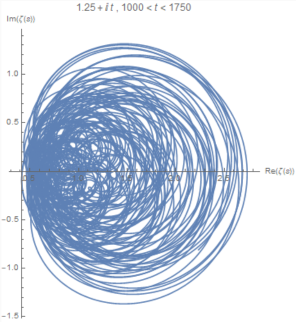
In order to study the image of vertical lines by the series (1) the condition that the exponents λ_n are linearly independent in the field of rational numbers has been assumed in [4]. As we suppose that $\lambda_1 = 0$, we cannot use the results in [4] for the series (1), yet we can study the series $f(s) = \sum_{n=2}^{\infty} a_n e^{-\lambda_n s}$ and translate the results obtained to $\zeta_{A,\Lambda}(s) = 1 + f(s)$. By [4], under the condition of linear independence of the exponents of f(s), the closure of the image by f(s) of the line $\mathrm{Re}\, s = \sigma_0$, where σ_0 is greater than the abscissa of absolute convergence of the series (1) (which is obviously the same as that of f(s)) is either a ring domain $r \le |z| \le R$ or a disc $|z| \le R$, according to the case where the sequence $\{|a_n|e^{-\lambda_n \sigma_0}\}$, $n \ge 2$ has a leading (vorhanden) term or not. Consequently, under the same condition, this image by $\zeta_{A,\Lambda}(s)$ will be a ring domain $r \le |z-1| \le R$, respectively a disc $|z-1| \le R$. We say that the numeric series $\sum_{n=1}^{\infty} \varrho_n$ has the leading term ϱ_{n_0} if $\varrho_{n_0} > \sum_{\rho_n \neq \varrho_{n_0}} \varrho_n$. The previous numbers R and r are respectively $R = \sum_{n=2}^{\infty} \varrho_n$ and $r = 2\varrho_{n_0} - R$, where $\varrho_n = |a_n|e^{-\lambda_n \sigma_0}$.

The results of Bohr are in agreement with the fact that the function (1) tends uniformly with respect to t to 1 as $\sigma \to +\infty$. Indeed, for $\sigma > \sigma_0$, where σ_0 is big enough, if there is a leading term ρ_{n_1} of the series $R = \sum_{n=2}^{\infty} |a_n| e^{-\lambda_n \sigma}$ then the image by $\zeta_{A,\Lambda}(s)$ of the line $\operatorname{Re} s = \sigma$, $\sigma \geq \sigma_0$ is included in the ring domain $|z - 1| \leq R = \zeta_{|A|,\Lambda}(\sigma) - 1$ and it is a dense set in this domain. Here we have denoted by $|A|$ the sequence $(|a_n|)$. As σ_0 gets smaller, some other term ρ_{n_2} can become leading term and then the image by $\zeta_{A,\Lambda}(s)$ of the line $\operatorname{Re} s = \sigma_0$ will be included in the ring domain $|z - \rho_{n_2}| \leq R$. If no leading term appears, then the respective ring domain will evolve into a disc. At the limit, as $\sigma_0 = \sigma_a$ we have $R = \infty$ and the image of the line $\operatorname{Re} s = \sigma_a$ is a dense set either in the whole complex plane or outside of an open disc. Hence we can state

Theorem

For any Dirichlet function $\zeta_{A,\Lambda}(s)$ the image of the line $\mathrm{Re}\, s = \sigma_0$, $\sigma_0 > \sigma_a$ is a dense set in either a ring domain $r \le |z-1| \le R$ or a disc $|z-1| \le R$ according to the fact that $R = \zeta_{|A|,\Lambda}(\sigma_0) - 1 = \sum_{n=2}^{\infty} |a_n| e^{-\lambda_n \sigma_0}$ has a leading term or not. When $\sigma_0 = \sigma_a$ we have $R = \infty$ and these domains become the exterior of an open disc or respectively the whole complex plane.

For the Riemann series, the term $\dfrac{1}{2^\sigma}$ is leading term for $\sum_{n=2}^{\infty} \frac{1}{n^\sigma}$ as long as $\zeta(\sigma) < 1 + \dfrac{1}{2^{\sigma-1}}$. Let us denote by σ_* the solution of the equation $\zeta(\sigma) = 1 + \dfrac{1}{2^{\sigma-1}}$. By inspecting a table of values of $\zeta(\sigma)$ we noticed that $2 < \sigma_* < 4$, hence for $1 < \sigma_0 \leq 2$ the image by $\zeta(s)$ of the line $\mathrm{Re}\, s = \sigma_0$ is included in the disc $|z-1| \leq R = \zeta(\sigma_0) - 1$ and it is a dense set in this disc. For $\sigma_0 \geq 4$, if we denote $r = \dfrac{1}{2^{\sigma_0 - 1}} - R$ then this image is included in the ring domain $r \leq |z-1| \leq R$.



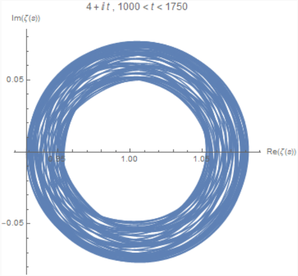


Figure 3: two situations for $\sigma_0 = 1.25$, respectively $\sigma_0 = 4$.

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